q-alg/9509xxxxx
ITEP-TH-6/95



# The Modified $q$-Bessel Functions and the $q$-Bessel-Macdonald Functions

M.A.Olshanetsky [1]
ITEP,117259, Moscow, Russia
e-mail olshanez@vxdesy.desy.de

V.-B.K.Rogov [2]
MIIT,101475, Moscow, Russia
e-mail m10106@sucemi.bitnet

September 1995

**Abstract**

We define a $q$-analog of the modified Bessel and Bessel-Macdonald functions. As for the q-Bessel functions of Jackson there is a couple of functions of the both kind. They are arisen in the Harmonic analysis on quantum symmetric spaces similarly to their classical counterpart. Their definition is based on the power expansions. We derive the recurrence relations, difference equations, q-Wronskians, and an analog of asymptotic expansions which turns out is exact in some domain if $q \neq 1$. Some relations for the basic hypergeometric function which follow from this fact are discussed.

## 1 Introduction

The q-analogues of the Bessel functions introduced ninety years ago by Jackson [1] are a subject of investigations in the last years [2, 3]. In these works their properties are derived in connection with the representation theory of quantum groups as well as their classical counterparts [4].

Our main interest is the q-analogues of the Bessel-Macdonald function $K_\nu$ (BMF) and modified Bessel functions $I_\nu$ (MBF). Their presence in the Harmonic analysis on homogenious spaces is at least threefold. First of all, BMF is the essential part of the Green function for the Laplace-Beltrami operator on noncompact symmetric spaces with nonpositive curvature [5, 6]. Next BMF define irreducible representations of the isometries of the pseudoEuclidean plane [4]. Finally, BMF are the simplest so-called Whittaker [7, 8, 9] functions, related to the symmetric spaces of rank one. It turns out that the Whittaker functions up to a gauge transform coincide with the wave functions of the open Toda quantum mechanical system [9]. In particular, for a rank one symmetric spaces they defined the wave functions of two-dimensional gravity in the minisuperspace approximation [10]. In both constructions MBF play an auxiliary role .

In fact the main motivation of our work is the last approach. In our previous work [11] we have described the Whittaker function for the quantum Lobachevsky space. They emerge as some special eigenfunctions of the quantum second Casimir operator. In this context it was possible to reduce the problem to commutative analysis, though the original formulation

[1] Supported by ISF-MIF-300 , RFFI-94-02-14365 and INTAS-93-0166 grants
[2] Supported by ISF grant JC 7100

is essentially noncommutative. The further progress in the Harmonic analysis demands to take into account the noncommutativity of the group algebra. It concerns, in particular, with the Poisson kernel for the Dirichlet problem in the quantum Lobachevsky space. In the classical case the Whittaker functions or, what is the same, BMF are the Fourier transform of the Poisson kernel. To reproduce this construction in the quantum case it is necessary to derive the most fundamental properties of q-MBF.

As in the classical case we begin with definition of q-MBF as the power expansions. There are two q-MBF and they are related to q-Bessel functions of Jackson [1] as the classical ones. We derive the action of difference operators, recurrence relation, difference equation and q-Wronskian for them. The most of these results can be easily derived from [2, 3], where q-Bessel functions were investigated. While the definition of q-MBF, based on the Jackson q-Bessel functions is strightforward, the definition of q-BMF is a rather subtle. We choose it in a such way that it becomes a holomorphic in the complex right half plane. We repeat the same program for q-BMF as for q-MBF. The most important part for the applications is the Laurent type expansion which is only asymptotics in the classical case and is represented by an convergent series in some domain in the quantum situation. Using this property we obtain in conclusion some relations for the basic hypergeometric function.

## 2 Some preliminary relations

We derive some needed relations for the fundamental functions $e_q(z), E_q(z)$, and $\Gamma_q(z)$ [12]. As by product we define q-psi function.

**1.**For an arbitrary $a$

$$(a,q)_n = \begin{cases} 1 & for \quad n \;=\; 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & for \quad n \;\geq\; 1, \end{cases}$$

$$(a,q)_\infty = \lim_{n\to\infty}(a,q)_n.$$

The $q$-exponentials are determined by formulas:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q,q)_n} = \frac{1}{(z,q)_\infty}, \qquad |z|<1, \tag{1}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{(q,q)_n} = (-z,q)_\infty, \qquad . \tag{2}$$

They are the q-deformations of the ordinary exponent

$$\lim_{q\to 1-0} e_q((1-q)z) = \lim_{q\to 1-0} E_q((1-q)z) = e^z.$$

Obviously, we have from (1) and (2)

$$e_q(z)E_q(-z) = 1 \tag{3}$$

It follows from (1) that the function $e_q(z)$ has the ordinary poles in the points $z = q^{-k}, k = 0, 1, \ldots$.

**Proposition 2.1 (F.I.Karpelevich)** *The $q$-exponential $e_q(z)$ can be represented as the sum of the partial functions*

$$e_q(z) = \frac{1}{(q,q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{(q,q)_k (1-zq^k)}. \tag{4}$$

**Proof.** Let

$$e_q(z,n) = \frac{1}{(q,q)_n} = \sum_{k=0}^{n} \frac{c_{k,n}}{1-zq^k}.$$

Then

$$c_{k,n} = res_{z=q^{-k}} e_q(z,n) = \lim_{z\to q^{-k}} (1-zq^k) e_q(z,n) = \frac{1}{(1-q^{-k})\cdots(1-q^{-1})(1-q)\dots(1-q^{n-k})} =$$

$$= \frac{(-1)^k q^{k(k+1)/2}}{(q,q)_k (q,q)_{n-k}}.$$

From this we have

$$e_q(z,n) = \sum_{k=0}^{n} \frac{(-1)^k q^{k(k+1)/2}}{(q,q)_k (1-zq^k)} \frac{1}{(q,q)_{n-k}}.$$

As $\frac{1}{(q,q)_{n-k}} < \frac{1}{(q,q)_\infty}$ we obtain

$$e_q(z) = \lim_{n\to\infty} e_q(z,n) = \frac{1}{(q,q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{(q,q)_k (1-zq^k)}. \blacksquare$$

The next Proposition follows from (1).

**Proposition 2.2**

$$\partial_q e_q((1-q)z) = e_q((1-q)z),$$

*where*

$$\partial_q f(z) = \frac{f(z)-f(qz)}{(1-q)z}. \tag{5}$$

The following Corollaries are evident.

**Corollary 2.1**

$$e_q(\frac{1-q^2}{2} qz) = \left(1 - \frac{1-q^2}{2} z\right) e_q(\frac{1-q^2}{2} z), \tag{6}$$

$$e_q(\frac{1-q^2}{2} q^{-1} z) = \frac{1}{1-\frac{q^{-1}(1-q^2)}{2} z} e_q(\frac{1-q^2}{2} z). \tag{7}$$

**Corollary 2.2**

$$e_{q^2}(\frac{(1-q^2)^2}{4} q^2 z^2) = \left[1 - \frac{(1-q^2)^2}{4} z^2\right] e_{q^2}(\frac{(1-q^2)^2}{4} z^2). \tag{8}$$

**Remark 2.1** *It follows from(1)*

$$e_q(\frac{1-q^2}{2} z) e_q(-\frac{1-q^2}{2} z) = \frac{1}{(\frac{1-q^2}{2} z, q)_\infty} \frac{1}{(\frac{1-q^2}{2} z, q)\infty} = \frac{1}{(\frac{(1-q^2)^2}{4} z^2, q^2)_\infty} = e_q^2(\frac{(1-q^2)^2}{4} z^2),$$

**2.** The $q$-gamma-function

$$\Gamma_q(\alpha) = \frac{q,q)_\infty}{(q^\alpha,q)\infty}(1-q)^{1-\alpha}. \tag{9}$$

If $n$ is a natural number then

$$\Gamma_q(n+1) = \frac{(q,q)_n}{(1-q)^n}.$$

**Proposition 2.3** *The function $\Gamma_{q^2}(z)$ can be represanted in the following form*

$$\Gamma_{q^2}(z) = (1-q^2)^{1-z}\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+1)}}{(q^2,q^2)_k(1-q^{2k+2z})}. \tag{10}$$

**Proof.** It follows from (9) that

$$\Gamma_{q^2}(z) = (q^2,q^2)_\infty(1-q^2)^{1-z}e_{q^2}(q^{2z}).$$

Substituting in this equality (4) we obtain (10). ■

We denote by $\psi_q(z)$ the logarithmic derivative of $\Gamma_q$-function

$$\psi_q(z) = \frac{\Gamma_q'(z)}{\Gamma_q(z)}. \tag{11}$$

From (9) and (11) we receve immediatly

**Proposition 2.4** *The function $\psi_{q^2}(z)$ has the following form*

$$\psi_{q^2}(z) = -\ln(1-q^2) + \ln q^2\sum_{k=1}^{\infty}\frac{q^{2k+2z}}{1-q^{2k+2z}}. \tag{12}$$

**Corollary 2.3**

$$\lim_{z\to -n}\frac{\psi_{q^2}(z)}{\Gamma_{q^2}(z)} = (-1)^n\frac{q^{-n(n+1)}(q^2,q^2)_n}{(1-q^2)^{n+1}}\ln q^2. \tag{13}$$

**Proof.** This result is obtained from (10) and (12). ■

# 3 The Modify $q$-Bessel Functions

**1. Definition.**
In [1] the $q$-Bessel functions were determined as

$$J_\nu^{(1)}(z,q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}(z/2)^\nu \;\; {}_2\Phi_1(0,0;q^{\nu+1};q,-\frac{z^2}{4}), \tag{14}$$

$$J_\nu^{(2)}(z,q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}(z/2)^\nu \;\; {}_0\Phi_1(-;q^{\nu+1};q,-\frac{z^2q^{\nu+1}}{4}), \tag{15}$$

where ${}_2\Phi_1$ and ${}_0\Phi_1$ are the basic hypergeometric functions [12]

$${}_r\Phi_s(a_1,\cdots,a_r;b_1,\cdots,b_s;q,z) = \sum_{n=0}^{\infty}\frac{(a_1,q)_n\ldots(a_r,q)_n}{(q,q)_n(b_1,q)_n\ldots(b_s,q)_n}[(-1)^n q^{n(n-1)/2}]^{1+s-r}z^n. \tag{16}$$

It allows to define q-MBF in analogy with the classical case [13] as (14) and (15).

**Definition 3.1** *The functions*

$$I_\nu^{(1)}(z,q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}(z/2)^\nu \ {}_2\Phi_1(0,0;q^{\nu+1};q,\frac{z^2}{4}),$$

$$I_\nu^{(2)}(z,q) = \frac{(q^{\nu+1},q)_\infty}{(q,q)_\infty}(z/2)^\nu \ {}_0\Phi_1(-;q^{\nu+1};q,\frac{z^2q^{\nu+1}}{4})$$

*are called the modify q-Bessel functions.*

Evidently,

$$I_\nu^{(j)}(z,q) = e^{-\frac{i\nu\pi}{2}}J_\nu^{(j)}(e^{i\pi/2}z,q), \qquad j=1,2.$$

We will consider below the functions

$$I_\nu^{(1)}((1-q^2)z;q^2) = \sum_{k=0}^{\infty}\frac{(1-q^2)^k(z/2)^{\nu+2k}}{(q^2,q^2)_k\Gamma_{q^2}(\nu+k+1)}, \qquad |z| < \frac{2}{1-q^2}, \tag{17}$$

$$I_\nu^{(2)}((1-q^2)z;q^2) = \sum_{k=0}^{\infty}\frac{q^{2k(\nu+k)}(1-q^2)^k(z/2)^{\nu+2k}}{(q^2,q^2)_k\Gamma_{q^2}(\nu+k+1)}. \tag{18}$$

If $|q| < 1$, the series (18) converges for all $z \neq 0$ absolutely. Therefore $I_\nu^{(2)}((1-q^2)z;q^2)$ is holomorphic function outside of $z=0$.

**Remark 3.1**

$$\lim_{q\to 1-0} I_\nu^{(j)}((1-q^2)z;q^2) = I_\nu(z), \qquad j=1,2.$$

**2. Recurrence relations and differnces.**

**Proposition 3.1** *The function $I_\nu^{(1)}((1-q^2)z;q^2)$ satisfies the following relations*

$$\frac{2}{(1+q)z}\partial_q z^\nu I_{-\nu}^{(1)}((1-q^2)z;q^2) = z^{\nu-1}I_{-\nu+1}^{(1)}((1-q^2)z;q^2), \tag{19}$$

$$\frac{2}{(1+q)z}\partial_q z^\nu I_{\nu}^{(1)}((1-q^2)z;q^2) = z^{\nu-1}I_{\nu-1}^{(1)}((1-q^2)z;q^2), \tag{20}$$

*where the operator $\partial_q$ is determined by (5).*

**Proof.**

$$\frac{2}{(1+q)z}\partial_q z^\nu I_{-\nu}^{(1)}((1-q^2)z;q^2) = \sum_{k=1}^{\infty}\frac{(1-q^2)^{k-1}z^{2k-2}}{2^{-\nu+2k-1}(q^2,q^2)_{k-1}\Gamma_{q^2}(-\nu+k+1)} =$$

$$= \sum_{k=0}^{\infty}\frac{(1-q^2)^k z^{2k}}{2^{-\nu+1+2k}(q^2,q^2)_k\Gamma_{q^2}(-\nu+1+k+1)} = z^{\nu-1}I_{-\nu+1}((1-q^2)z;q^2).$$

The proof of (20) is the same.■

**Proposition 3.2** *The function* $I_\nu^{(1)}((1-q^2)z;q^2)$ *satisfies the following recurrence relations*

$$I_{\nu-1}^{(1)}((1-q^2)z;q^2) - I_{\nu+1}^{(1)}((1-q^2)z;q^2) = \frac{2}{(1-q^2)z}(q^{-\nu}-q^{\nu})I_\nu^{(1)}((1-q^2)qz;q^2), \tag{21}$$

$$I_{\nu-1}^{(1)}((1-q^2)z;q^2) + I_{\nu+1}^{(1)}((1-q^2)z;q^2) = \frac{4}{(1-q^2)z}I_\nu^{(1)}((1-q^2)z;q^2) -$$
$$-\frac{2}{(1-q^2)z}(q^{-\nu}+q^{\nu})I_\nu^{(1)}((1-q^2)qz;q^2). \tag{22}$$

**Proof.** It follows from (19) and (20)

$$I_{\nu-1}^{(1)}((1-q^2)z;q^2) = \frac{2}{(1-q^2)z}\left[I_\nu^{(1)}((1-q^2)z;q^2) - q^{\nu}I_\nu^{(1)}((1-q^2)qz;q^2)\right],$$

$$I_{\nu+1}^{(1)}((1-q^2)z;q^2) = \frac{2}{(1-q^2)z}\left[I_\nu^{(1)}((1-q^2)z;q^2) - q^{-\nu}I_\nu^{(1)}((1-q^2)qz;q^2)\right].$$

Summing and subtracting these equalities we obtain the statement .■

**Proposition 3.3** *The function* $I_\nu^{(2)}((1-q^2)z;q^2)$ *satisfies the following relations*

$$\frac{2}{(1+q)z}\partial_q z^{\nu} I_{-\nu}^{(2)}((1-q^2)z;q^2) = q^{-\nu+1}z^{\nu-1}I_{-\nu+1}^{(2)}((1-q^2)qz;q^2),$$

$$\frac{2}{(1+q)z}\partial_q z^{\nu} I_{\nu}^{(2)}((1-q^2)z;q^2) = q^{-\nu+1}z^{\nu-1}I_{\nu-1}^{(2)}((1-q^2)qz;q^2).$$

**Proposition 3.4** *The function* $I_\nu^{(2)}((1-q^2)z;q^2)$ *satisfies the following recurrence relations*

$$q^{-\nu}I_{\nu-1}^{(2)}((1-q^2)z;q^2) - q^{\nu}I_{\nu+1}^{(2)}((1-q^2)z;q^2) = \frac{2}{(1-q^2)z}(q^{-\nu}-q^{\nu})I_\nu^{(2)}((1-q^2)z;q^2),$$

$$q^{-\nu}I_{\nu-1}^{(2)}((1-q^2)z;q^2) + q^{\nu}I_{\nu+1}^{(2)}((1-q^2)z;q^2) = \frac{4}{(1-q^2)z}I_\nu^{(2)}((1-q^2)q^{-1}z;q^2) -$$
$$\frac{2}{(1-q^2)z}(q^{-\nu}+q^{\nu})I_\nu^{(2)}((1-q^2)z;q^2).$$

The proof of Propositions 3.3 and 3.4 are the same as 3.1 and 3.2.

**3. Difference equation.**

**Proposition 3.5** *The function* $I_\nu^{(1)}((1-q^2)z;q^2)$ *is a solution to the difference equation*

$$\left[1-\frac{q^{-2}(1-q^2)^2}{4}z^2\right]f(q^{-1}z) - (q^{-\nu}+q^{\nu})f(z) + f(qz) = 0. \tag{23}$$

**Proof.** Substituting (17) in the left side of (23) we obtain

$$\sum_{k=0}^{\infty}\left[q^{-\nu-2k} - q^{-\nu} - q^{\nu} + q^{\nu+2k} - \frac{q^{-\nu-2k-2}(1-q^2)^2z^2}{4}\right]\frac{(1-q^2)^k(z/2)^{\nu+2k}}{(q^2,q^2)_k\Gamma_{q^2}(\nu+k+1)} =$$

$$= \sum_{k=1}^{\infty}\frac{q^{-\nu-2k}(1-q^{2k})(1-q^{2\nu+2k})(1-q^2)^k(z/2)^{\nu+2k}}{(q^2,q^2)_k\Gamma_{q^2}(\nu+k+1)} - \sum_{k=0}^{\infty}\frac{q^{-\nu-2k-2}(1-q^2)^{k+2}(z/2)^{\nu+2k+2}}{(q^2,q^2)_k\Gamma_{q^2}(\nu+k+1)}.$$

Changing $k$ to $k+1$ in the first term we obtain zero.■

**Corollary 3.1** *The function $I^{(1)}_{-\nu}((1-q^2)z;q^2)$ satisfies the equation (23).*

**Proof.** Obviously the left side of (23) is a even function of $\nu$. Then the change $\nu$ to $-\nu$ in (17) transforms a solution to (23) to a solution.■

**Definition 3.2** *The $q$-Wronskian of two solutions $f^1_\nu(z)$, $f^2_\nu(z)$ to a difference second order equation is*

$$W(f^1_\nu,f^2_\nu)(z)=f^1_\nu(z)f^2_\nu(qz)-f^1_\nu(qz)f^2_\nu(z).$$

If $q$-Wronskian nonvanishes an arbitrary solution to the difference second order equation can be written as

$$f_\nu(z)=C_1f^1_\nu(z)+C_2f^2_\nu(z).$$

In this case the functions $f^1_\nu(z)$ and $f^2_\nu(z)$ is the fundamential system of solutions.

**Proposition 3.6** *If $\nu$ is noninteger the functions $I^{(1)}_\nu((1-q^2)z;q^2)$ and $I^{(1)}_{-\nu}((1-q^2)z;q^2)$ form a fundamental system of solutions to the equation (23).*

**Proof.** Consider the $q$-Wronskian

$$W(I^{(1)}_\nu,I^{(1)}_{-\nu})(z)=$$

$$I^{(1)}_\nu((1-q^2)z;q^2)I^{(1)}_{-\nu}((1-q^2)qz;q^2)-I^{(1)}_\nu((1-q^2)qz;q^2)I^{(1)}_{-\nu}((1-q^2)z;q^2). \tag{24}$$

Since $I^{(1)}_\nu$ and $I^{(1)}_{-\nu}$ are solution to (23) then

$$I^{(1)}_{\pm\nu}((1-q^2)q^2z;q^2)=-[1-\frac{(1-q^2)^2}{4}z^2]I^{(1)}_{\pm\nu}((1-q^2)z;q^2)+(q^{-\nu}+q^\nu)I^{(1)}_{\pm\nu}((1-q^2)qz;q^2).$$

Thus

$$W(I^{(1)}_\nu,I^{(1)}_{-\nu})(qz)=I^{(1)}_\nu((1-q^2)qz;q^2)I^{(1)}_{-\nu}((1-q^2)q^2z;q^2)-$$

$$I^{(1)}_\nu((1-q^2)q^2z;q^2)I^{(1)}_{-\nu}((1-q^2)qz;q^2)=[1-\frac{(1-q^2)^2}{4}z^2]W(I^{(1)}_\nu,I^{(1)}_{-\nu})(z).$$

Comparing this equality with (8) we can write

$$W(I^{(1)}_\nu,I^{(1)}_{-\nu})(z)=C_\nu e_{q^2}(\frac{(1-q^2)^2}{4}z^2).$$

Setting $z=0$ in (24) we obtain

$$C_\nu=\frac{q^{-\nu}(1-q^2)}{\Gamma_{q^2}(\nu)\Gamma_{q^2}(1-\nu)}.$$

So finaly

$$W(I^{(1)}_\nu,I^{(1)}_{-\nu})(z)=\frac{q^{-\nu}(1-q^2)}{\Gamma_{q^2}(\nu)\Gamma_{q^2}(1-\nu)}e_{q^2}(\frac{(1-q^2)^2}{4}z^2). \tag{25}$$

Obviously this function nonvanishes.■

If $\nu=n$ is integer then from (17) and (18)

$$I^{(j)}_{-n}((1-q^2)z;q^2)=I^{(j)}_n((1-q^2)z;q^2),\quad j=1,2. \tag{26}$$

**Proposition 3.7** *The function* $I_\nu^{(2)}((1-q^2)z;q^2)$ *is a solution to the equation*

$$f(q^{-1}z)-(q^{-\nu}+q^{\nu})f(z)+\left[1-\frac{(1-q^2)^2}{4}z^2\right]f(qz)=0. \tag{27}$$

This Proposition is proved in the same way as Proposition 3.5.

**Corollary 3.2** *The q-MBF are related as (17) and (18)*

$$I_\nu^{(1)}((1-q^2)z;q^2)=e_{q^2}(\frac{(1-q^2)^2}{4}z^2)I_\nu^{(2)}((1-q^2)z;q^2) \tag{28}$$

**Proof.** It follows from (27) and (8) that

$$e_{q^2}(\frac{(1-q^2)^2}{4}z^2)I_\nu^{(2)}((1-q^2)z;q^2)$$

satisfies (23). Hence if $\nu$ is noninteger

$$e_{q^2}(\frac{(1-q^2)^2}{4}z^2)I_\nu^{(2)}((1-q^2)z;q^2)=AI_\nu^{(1)}((1-q^2)z;q^2)+BI_{-\nu}^{(1)}((1-q^2)z;q^2). \tag{29}$$

Multiplying this equality on $(z/2)^\nu$ and setting $z=0$ we obtain $B=0$. Multiplying (29) on $(z/2)^{-\nu}$ and setting $z=0$ we obtain $A=1$.

Since (17) and (18) are continuous functions of $\nu$, then (28) is valid for $\nu=n$ as well.$\blacksquare$

Multiplying the both sides of (28) on $E_{q^2}(-\frac{(1-q^2)^2}{4}z^2)$ we obtain

**Corollary 3.3**

$$I_\nu^{(2)}((1-q^2)z;q^2)=E_{q^2}(-\frac{(1-q^2)^2}{4}z^2)I_\nu^{(1)}((1-q^2)z;q^2).$$

**Proposition 3.8** *The function* $I_\nu^{(1)}((1-q^2)z;q^2)$ *is the meromorphic function outside of* $z=0$ *with the ordinary poles in the points* $z=\pm\frac{2q^{-r}}{1-q^2}, r=0,1,\dots$

**Proof.** The statement of this Proposition follows from (28) and Remark 2.1 as

$$e_{q^2}(\frac{(1-q^2)^2}{4}z^2)=\frac{1}{(\frac{(1-q^2)^2}{4}z^2,q^2)_\infty}=\frac{1}{(\frac{(1-q^2)}{2}z,q)_\infty(-\frac{(1-q^2)}{2}z,q)_\infty}=$$

$$=e_q(\frac{1-q^2}{2}z)e_q(-\frac{1-q^2}{2}z).\blacksquare$$

**Remark 3.2** *If* $q\to 1-0$ *the all poles of* $I_\nu^{(1)}((1-q^2)z;q^2)$

$$z_r=\pm\frac{2q^{-r}}{1-q^2}, \qquad r=0,1,\dots$$

*go to infinity along the real axis.*

## 4 The Laurent Type Serieses for Modify $q$-Bessel Functions

Unfortunately the function $I_\nu^{(1)}((1-q^2)z;q^2)$ is determined by power series (17) in domain $|z|<\frac{2}{1-q^2}$ only while we need a representation of this function as series on the whole complex plane. Now we will try to improve this situation.

**Proposition 4.1** *An arbitrary solution to (23) can be written in form*

$$f_\nu(z)=\frac{a}{\sqrt{z}}e_q(\frac{1-q^2}{2}z)\ {}_2\Phi_1(q^{\nu+1/2},q^{-\nu+1/2};-q;q,\frac{2q}{(1-q^2)z})+$$

$$+\frac{b}{\sqrt{z}}e_q(-\frac{1-q^2}{2}z)\ {}_2\Phi_1(q^{\nu+1/2},q^{-\nu+1/2};-q;q,-\frac{2q}{(1-q^2)z}), \tag{30}$$

*where ${}_2\Phi_1$ is determined by (16).*

**Proof.** Let $f_\nu(z)$ be a solution to (23). Represent it as

$$f_\nu(z)=\frac{a}{\sqrt{z}}e_q(\frac{1-q^2}{2}z)\phi^1_\nu(z)+\frac{b}{\sqrt{z}}e_q(-\frac{1-q^2}{2}z)\phi^2_\nu(z). \tag{31}$$

We assume here that the first summand on the right side satisfyes the equation (23). Then the second summand is also a solution. Consider the first summand. Substituting it in (23) and using (6), (7) we obtain the difference equation for $\phi^1_\nu(z)$

$$\left[1+\frac{q^{-1}(1-q^2)}{2}z\right]q\phi^1_\nu(q^{-1}z)-q^{1/2}(q^{-\nu}+q^\nu)\phi^1_\nu(z)+\left[1-\frac{1-q^2}{2}z\right]\phi^1_\nu(qz)=0. \tag{32}$$

Represent $\phi^1_\nu(z)$ as

$$\phi^1_\nu(z)=\sum_{k=0}^\infty a_k z^{-k}.$$

Then $a_0$ is arbitrary, and for any $k\ge 1$

$$a_k=a_{k-1}\frac{2q(1-q^{\nu-1/2+k})(1-q^{-\nu-1/2+k})}{(1-q^2)(1-q^{2k})}.$$

and

$$a_k=a_0\frac{2^kq^k(q^{\nu+1/2},q)_k(q^{-\nu+1/2},q)_k}{(1-q^2)^k(q^2,q^2)_k}.$$

Assuming $a_0=1$ we obtain

$$\phi^1_\nu(z)=\sum_{k=0}^\infty\frac{2^kq^k(q^{\nu+1/2},q)_k(q^{-\nu+1/2},q)_k}{(1-q^2)^k(q^2,q^2)_k}z^{-k}. \tag{33}$$

Similarly

$$\phi^2_\nu(z)=\sum_{k=0}^\infty(-1)^k\frac{2^kq^k(q^{\nu+1/2},q)_k(q^{-\nu+1/2},q)_k}{(1-q^2)^k(q^2,q^2)_k}z^{-k}=\phi^1_\nu(-z).$$

In view of this fact we will drop the superscript. The series (33) converges absolutly for $|z|>\frac{2q}{1-q^2}$.

The coefficients $a$ and $b$ are determined uniquely by $f_\nu(z)$. In fact let $z = z_0, |z_0| > 2/(1-q^2), f_\nu(z_0) = A$ and $f_\nu(qz_0) = B$. Using (6), (7) we come to the system for $a$ and $b$:

$$\begin{cases} \frac{a}{\sqrt{z_0}} e_q(\frac{1-q^2}{2}z)\phi_\nu(z_0) + \frac{b}{\sqrt{z_0}} e_q(-\frac{1-q^2}{2}z)\phi_\nu(-z_0) & = A \\ \frac{a}{\sqrt{qz_0}}(1-(1-q^2)/2z_0)e_q(\frac{1-q^2}{2}z)\phi_\nu(qz_0) + \frac{b}{\sqrt{qz_0}}(1+(1-q^2)/2z_0)e_q(-\frac{1-q^2}{2}z)\phi_\nu(-qz_0) & = B. \end{cases} \tag{34}$$

Its determinant has the form

$$W = \frac{q^{-1/2}}{z_0} e_{q^2}(\frac{(1-q^2)^2}{4}z^2)[\phi_\nu(z_0)\phi_\nu(-qz_0) - \phi_\nu(qz_0)\phi_\nu(-z_0)+$$

$$+\frac{1-q^2}{2}z_0(\phi_\nu(z_0)\phi_\nu(-qz_0) + \phi_\nu(qz_0)\phi_\nu(-z_0))].$$

Assume for a moment that $W(z_0) = 0$ for some $z_0 : |z| > \frac{2}{1-q^2}$. Then

$$\begin{cases} \phi_\nu(z_0)\phi_\nu(-qz_0) - \phi_\nu(qz_0)\phi_\nu(-z_0) = 0 \\ \phi_\nu(z_0)\phi_\nu(-qz_0) + \phi_\nu(qz_0)\phi_\nu(-z_0) = 0 \end{cases}$$

or

$$\begin{cases} \phi_\nu(z_0)\phi_\nu(-qz_0) = 0 \\ \phi_\nu(qz_0)\phi_\nu(-z_0) = 0 \end{cases} \tag{35}$$

It follows from (33) that (35) is fulfild if $\phi_\nu(z_0) = \phi_\nu(qz_0) = 0$ (or $\phi_\nu(-z_0) = \phi_\nu(-qz_0) = 0$). In this case from (32) we have $\phi_\nu(q^{-1}z_0) = 0$. And hence $\phi_\nu(q^{-r}z_0) = 0$ for any $r = 0, 1, \ldots$. But it contrudicts the obvious equality $\lim_{|z|\to\infty} \phi_\nu(z) = 1$. Thus $W \neq 0$, and $a$, $b$ are determined uniquely from (34).

It is easy to see from (16) that

$$\phi_\nu(z) = \sum_{k=0}^{\infty} \frac{(q^{\nu+1/2}, q)_k (q^{-\nu+1/2}, q)_k}{(q, q)_k (-q, q)_k} \left(\frac{2q}{(1-q^2)z}\right)^k = {}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, \frac{2q}{(1-q^2)z}).$$

Substituting this function to (31) we obtain (30). ■

We denote

$${}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, \frac{2q}{(1-q^2)z}) = \Phi_\nu(z) \tag{36}$$

for brevity.

**Proposition 4.2** *The q-MBF $I_\nu^{(1)}((1-q^2)z; q^2)$ for $z \neq 0$ can be represented as*

$$I_\nu^{(1)}((1-q^2)z; q^2) = \frac{a_\nu}{\sqrt{z}} \left[ e_q(\frac{1-q^2}{2}z)\Phi_\nu(z) + ie^{i\nu\pi} e_q(-\frac{1-q^2}{2}z)\Phi_\nu(-z) \right], \tag{37}$$

*where $\Phi_\nu(z)$ is determined by (36) and*

$$a_\nu = \sqrt{\frac{2}{1-q^2}} e_q(-1) \frac{I_\nu^{(2)}(2; q^2)}{\Phi_\nu(\frac{2}{1-q^2})}. \tag{38}$$

**Proof.** It follows from corollary 3.2 and Proposition 4.1 that

$$e_q(\frac{1-q^2}{2}z)e_q(-\frac{1-q^2}{2}z)I_\nu^{(2)}((1-q^2)z;q^2)=$$

$$=\frac{a_\nu}{\sqrt{z}}e_q(\frac{1-q^2}{2}z)\Phi_\nu(z)+\frac{b_\nu}{\sqrt{z}}e_q(-\frac{1-q^2}{2}z)\Phi_\nu(-z) \tag{39}$$

in the domain $\frac{2q}{1-q^2}<|z|<\frac{2}{1-q^2}$. The functions in the right and the left sides are meromorphic in the domain $z\neq 0$, and have the ordinary poles in the points $z=\pm\frac{2q^{-r}}{1-q^2}, r=0,1,\cdots$. Due to the uniqueness of the analytic continuation the equality (39) is valid in the domain $z\neq 0$. We require that the residues in the poles of both sides (39) are equal. Then for $z=\frac{2q^{-r}}{1-q^2}$

$$e_q(-q^{-r})I_\nu^{(2)}(2q^{-r};q^2)=a_\nu q^{r/2}\sqrt{\frac{1-q^2}{2}}\Phi_\nu(\frac{2q^{-r}}{1-q^2}), \tag{40}$$

and for $z=-\frac{2q^{-r}}{1-q^2}$

$$e_q(-q^{-r})I_\nu^{(2)}(-2q^{-r};q^2)=b_\nu\frac{q^{r/2}}{i}\sqrt{\frac{1-q^2}{2}}\Phi_\nu(-\frac{2q^{-r}}{1-q^2}). \tag{41}$$

It follows from (18) $I_\nu^{(2)}(-2q^{-r};q^2)=e^{i\nu\pi}I_\nu(2q^{-r};q^2)$. Hence from (40) and (41)

$$b_\nu=ie^{i\nu\pi}a_\nu. \tag{42}$$

Assuming $r=0$ in (40) we have (38). Substituting (42) to (39) we obtain the statement of the Proposition. ■

**Proposition 4.3** *The coefficients $a_\nu$ (38) satisfy the recurrent relation*

$$a_{\nu+1}=a_\nu q^{-\nu-1/2}, \tag{43}$$

*and the condition*

$$a_\nu a_{-\nu}=\frac{q^{-\nu+1/2}}{2\Gamma_{q^2}(\nu)\Gamma_{q^2}(1-\nu)\sin\nu\pi}. \tag{44}$$

**Proof.** Substitute (37) in (21) and (22). Then using (6) we have

$$a_{\nu-1}\Phi_{\nu-1}(z)-a_{\nu+1}\Phi_{\nu+1}(z)=2a_\nu q^{-1/2}\frac{q^{-\nu}-q^\nu}{(1-q^2)z}(1-\frac{1-q^2}{2}z)\Phi_\nu(qz), \tag{45}$$

$$a_{\nu-1}\Phi_{\nu-1}(z)+a_{\nu+1}\Phi_{\nu+1}(z)=\frac{4a_\nu}{(1-q^2)z}\Phi_\nu(z)-$$

$$-2a_\nu q^{-1/2}\frac{q^{-\nu}+q^\nu}{(1-q^2)z}(1-\frac{1-q^2}{2}z)\Phi_\nu(qz). \tag{46}$$

Turn $z$ to the infinity in (45) and (46) we come to the system

$$\begin{cases} a_{\nu-1}-a_{\nu+1}=-a_\nu q^{-1/2}(q^{-\nu}-q^\nu) \\ a_{\nu-1}+a_{\nu+1}=a_\nu q^{-1/2}(q^{-\nu}+q^\nu). \end{cases}$$

From this system we obtain the first statement of the Proposition.

Consider the $q$-Wronskian $W(I_{\nu}^{(1)}, I_{-\nu}^{(1)})$ (25), and the representation (37). Then for $\nu \neq n$

$$\frac{q^{-\nu}(1-q^2)}{\Gamma_{q^2}(\nu)\Gamma_{q^2}(1-\nu)} e_{q^2}(\frac{(1-q^2)^2}{4} z^2) = q^{-1/2} a_{\nu} a_{-\nu} z^{-1} e_q(\frac{1-q^2}{2} z) e_q(-\frac{1-q^2}{2} z) \times$$

$$\times \{ i e^{-i\nu\pi} [(1 + \frac{1-q^2}{2} z) \Phi_{\nu}(z) \Phi_{\nu}(-qz) - (1 - \frac{1-q^2}{2} z) \Phi_{\nu}(qz) \Phi_{\nu}(-z)] +$$

$$+ i e^{i\nu\pi} [(1 - \frac{1-q^2}{2} z) \Phi_{\nu}(-z) \Phi_{\nu}(qz) - (1 + \frac{1-q^2}{2} z) \Phi_{\nu}(-qz) \Phi_{\nu}(z)] \}.$$

Reducing this equality to the $q$-exponentials and turning $z$ to infinity we obtain (44).■

If $\nu = n$ from (43) we have

$$a_n = a_{n-k} q^{-k/2(2n-k)}.$$

If $k = 2n$ then $a_n = a_{-n}$.

From (44) and [12] (1.10.16) we obtain

$$a_n^2 = \frac{q^{-n^2+1/2} \ln q^{-2}}{2\pi(1-q^2)}. \tag{47}$$

**Proposition 4.4** *The q-MBF $I_{\nu}^{(2)}((1-q^2)z; q^2)$ for $z \neq 0$ can be represented by*

$$I_{\nu}^{(2)}((1-q^2)z; q^2) = \frac{a_{\nu}}{\sqrt{z}} \left[ E_q(\frac{1-q^2}{2} z) \Phi_{\nu}(z) + i e^{i\nu\pi} E_q(-\frac{1-q^2}{2} z) \Phi_{\nu}(-z) \right]. \tag{48}$$

**Proof**. This statement follows from Corollary 3.3 and (3).

# 5 The $q$-Bessel-Macdonald Functions

In the classical analyses the BMF are defined as

$$K_{\nu}(z) = \frac{\pi}{2 \sin \nu\pi} [I_{-\nu}(z) - I_{\nu}(z)] \tag{49}$$

for $\nu \neq n$, and if $\nu = n$ by the limit for $\nu \to n$ in (49). Here we present the correct "quantization" of this definition in a such way that other properties are also quantized in a consistent way.

**Lemma 5.1** *For $a_{\nu}$ (38)*

$$(\frac{\partial}{\partial \nu} a_{\nu} - \frac{\partial}{\partial \nu} a_{-\nu})|_{\nu=n} = a_n \tilde{a}, \tag{50}$$

*where*

$$\tilde{a} = \frac{2 \ln q^2}{I_0^{(2)}(2; q^2)} \sum_{k=0}^{\infty} \frac{q^{2k^2}}{(q^2, q^2)_k^2} (k - \sum_{l=1}^{\infty} \frac{q^{2l+2k+2}}{1 - q^{2l+2k+2}}).$$

**Proof.** Let $n = [\nu]$. It follows from (43)

$$a_{\nu} = a_{\nu-n} q^{-n\nu + \frac{n^2}{2}}, \qquad a_{-\nu} = a_{-\nu+n} q^{-n\nu + \frac{n^2}{2}}.$$

Then

$$(\frac{\partial}{\partial\nu}a_\nu - \frac{\partial}{\partial\nu}a_{-\nu})|_{\nu=n} = (\frac{\partial}{\partial\nu}a_{\nu-n} - \frac{\partial}{\partial\nu}a_{-\nu+n})|_{\nu=n}q^{-\frac{n^2}{2}}.$$

Using (38) we obtain

$$(\frac{\partial}{\partial\nu}a_\nu - \frac{\partial}{\partial\nu}a_{-\nu})|_{\nu=0} =$$

$$= \frac{\sqrt{2/(1-q^2)}e_q(-1)I_0^{(2)}(2;q^2)}{\Phi_0(2/(1-q^2)}\frac{q^{-\frac{n^2}{2}}}{I_0^{(2)}(2;q^2)}\left(\frac{\partial}{\partial\nu}I_\nu^{(2)}(2;q^2) - \frac{\partial}{\partial\nu}I_{-\nu}^{(2)}(2;q^2)\right)|_{\nu=0} =$$

$$= a_0\frac{q^{-\frac{n^2}{2}}}{I_0^{(2)}(2;q^2)}\left(\frac{\partial}{\partial\nu}I_\nu^{(2)}(2;q^2) - \frac{\partial}{\partial\nu}I_{-\nu}^{(2)}(2;q^2)\right)|_{\nu=0}. \qquad (51)$$

It follows from (18)

$$\frac{\partial}{\partial\nu}I_\nu^{(2)}(2;q^2) = \ln q^2\sum_{k=0}^{\infty}\frac{kq^{2k(\nu+k)}}{(q^2,q^2)_k(1-q^2)^{\nu+k}\Gamma_{q^2}(\nu+k+1)} - \ln(1-q^2)I_\nu^{(2)}(2;q^2) -$$

$$-\sum_{k=0}^{\infty}\frac{q^{2k(\nu+k)}\psi_{q^2}(\nu+k+1)}{(q^2,q^2)_k(1-q^2)^{\nu+k}\Gamma_{q^2}(\nu+k+1)}.$$

From (12) and (13)

$$(\frac{\partial}{\partial\nu}I_\nu^{(2)}(2;q^2) - \frac{\partial}{\partial\nu}I_{-\nu}^{(2)}(2;q^2))|_{\nu=0} =$$

$$= 2\ln q^2\sum_{k=0}^{\infty}\frac{kq^{2k^2}}{(q^2,q^2)_k^2} - 2\ln(1-q^2)I_0^{(2)}(2;q^2) - 2\sum_{k=0}^{\infty}\frac{q^{2k^2}}{(q^2,q^2)_k^2}\psi_{q^2}(k+1) =$$

$$= 2\ln q^2\sum_{k=0}^{\infty}\frac{kq^{2k^2}}{(q^2,q^2)_k^2} - 2\ln q^2\sum_{k=0}^{\infty}\frac{q^{2k^2}}{(q^2,q^2)_k^2}\sum_{l=1}^{\infty}\frac{q^{2l+2k+2}}{1-q^2l+2k+2}. \qquad (52)$$

Now (50) follows from (51) and (52).

**Definition 5.1** *The q-Bessel-Macdonald functions (q-BMF) are defined as*

$$K_\nu^{(j)}((1-q^2)z;q^2) = \frac{q^{-\nu^2+1/2}}{4(a_\nu a_{-\nu})^{3/2}\sin\nu\pi}\left[a_\nu I_{-\nu}^{(j)}((1-q^2)z;q^2) - a_{-\nu}I_\nu^{(j)}((1-q^2)z;q^2)\right] \qquad (53)$$

*with $a_\nu$ (38), $j=1,2$.*

As in the classical case this definition should be adjusted for the integer values of the index $\nu = n$. Consider the limit of (53) for $j=1$ taking into account (47)

$$K_n^{(1)}((1-q^2)z;q^2) = \frac{q^{-n^2+1/2}}{4a_n^3}\lim_{\nu\to n}\frac{a_\nu I_{-\nu}^{(1)}((1-q^2)z;q^2) - a_{-\nu}I_\nu^{(1)}((1-q^2)z;q^2)}{\sin\nu\pi}.$$

Using (17) we can write

$$\frac{\partial}{\partial\nu}I_\nu^{(1)}((1-q^2)z;q^2) = \ln z/2I_\nu^{(1)}((1-q^2)z;q^2) - \sum_{k=0}^{\infty}\frac{(1-q^2)^k(z/2)^{\nu+k}\psi_{q^2}(\nu+k+1)}{(q^2,q^2)_k\Gamma_{q^2}(\nu+k+1)}.$$

Due to Corollary 2.1

$$[\frac{\partial}{\partial\nu}I^{(1)}_{-\nu}((1-q^2)z;q^2)-\frac{\partial}{\partial\nu}I^{(1)}_{\nu}((1-q^2)z;q^2)]|_{\nu=n}=-2\ln z/2I^{(1)}_n((1-q^2)z;q^2)+$$

$$+\ln q^2\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{q^{-(n-k-1)(n-k-2)}(1-q^2)^{-n+2k}(q^2,q^2)_{n-k-1}}{(q^2,q^2)_k}(z/2)^{-n+2k}+$$

$$+2\sum_{k=0}^{\infty}\frac{(1-q^2)^{n+2k}(z/2)^{n+2k}}{(q^2,q^2)_k(q^2,q^2)_{n+k}}[\psi_{q^2}(n+k+1)+\psi_{q^2}(k+1)]. \tag{54}$$

Thus

$$K^{(1)}_n((1-q^2)z;q^2)=\frac{(-1)^nq^{-n^2+1/2}}{4\pi a_n^3}[(\frac{\partial}{\partial\nu}a_\nu-\frac{\partial}{\partial\nu}a_{-\nu})|_{\nu=n}I^{(1)}_n((1-q^2)z;q^2)+$$

$$+a_n(\frac{\partial}{\partial\nu}I^{(1)}_{-\nu}((1-q^2)z;q^2)-\frac{\partial}{\partial\nu}I^{(1)}_{\nu}((1-q^2)z;q^2))|_{\nu=n}].$$

The final expression follows from Lemma 5.1 and (54) that

$$K^{(1)}_n((1-q^2)z;q^2)=\frac{(-1)^nq^{-n^2+1/2}}{4\pi a_n^2}\{(\tilde a-2\ln z/2)I^{(1)}_n((1-q^2)z;q^2)+$$

$$+\ln q^2\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{q^{-(n-k-1)(n-k-2)}(1-q^2)^{-n+2k}(q^2,q^2)_{n-k-1}}{(q^2,q^2)_k}(z/2)^{-n+2k}+$$

$$+\sum_{k=0}^{\infty}\frac{(1-q^2)^{n+2k}(z/2)^{n+2k}}{(q^2,q^2)_k(q^2,q^2)_{n+k}}\Big[\psi_{q^2}(n+k+1)+\psi_{q^2}(k+1)\Big]\}.$$

For $j=2$ we have

$$K^{(2)}_n((1-q^2)z;q^2)=\frac{(-1)^nq^{-n^2+1/2}}{4\pi a_n^2}\{(\tilde a-2\ln z/2)I^{(2)}_n((1-q^2)z;q^2)+$$

$$+\ln q^2\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{q^{-(n-k-1)(n-k-2)+2k(k-n)}(1-q^2)^{-n+2k}(q^2,q^2)_{n-k-1}}{(q^2,q^2)_k}(z/2)^{-n+2k}+$$

$$+\sum_{k=0}^{\infty}\frac{q^{2k(n+k)}(1-q^2)^{n+2k}(z/2)^{n+2k}}{(q^2,q^2)_k(q^2,q^2)_{n+k}}\Big[\psi_{q^2}(n+k+1)+\psi_{q^2}(k+1)\Big]\}.$$

**Proposition 5.1** *q-BMF $K^{(1)}_\nu((1-q^2)z;q^2)$ is a holomorphic function in the domain Re$z>\frac{2q}{1-q^2}$*

**Proof.** Substitute (37) in (53). Then

$$K^{(1)}_\nu((1-q^2)z;q^2)=\frac{q^{-\nu^2+1/2}}{2\sqrt{a_\nu a_{-\nu}}\sqrt{z}}e_q(-\frac{1-q^2}{2}z)\Phi_\nu(z). \tag{55}$$

The product of $e_q(-\frac{1-q^2}{2}z)$ and ${}_2\Phi_1(q^{\nu+1/2},q^{-\nu+1/2};-q;q,\frac{2q}{(1-q^2)z})$ is a holomorphic function in Re$z>\frac{2q}{1-q^2}$.

**Proposition 5.2** *q-BMF* $K_\nu^{(2)}((1-q^2)z;q^2)$ *is a holomorphic function in the domain* $z\neq 0$

**Proof.** This statement follows from (53) and (18).

**Corollary 5.1**

$$K_\nu^{(1)}((1-q^2)z;q^2)0=\frac{q^{-\nu^2+1/2}}{2\sqrt{a_\nu a_{-\nu}}\sqrt{z}}E_q(-\frac{1-q^2}{2}z)\Phi_\nu(z). \tag{56}$$

**Proof.** This formula follows from (53) and (48).

**Proposition 5.3** *The function* $K_\nu^{(1)}((1-q^2)z;q^2)$ *satisfies the following relations*

$$\frac{2}{(1+q)z}\partial_q z^\nu K_\nu^{(1)}((1-q^2)z;q^2)=-z^{\nu-1}K_{\nu-1}((1-q^2)z;q^2), \tag{57}$$

$$\frac{2}{(1+q)z}\partial_q z^{-\nu} K_\nu^{(1)}((1-q^2)z;q^2)=-z^{-\nu-1}K_{\nu+1}((1-q^2)z;q^2).$$

**Proof.** It follows from Definition 5.1 and Proposition 3.1

$$\frac{2}{(1+q)z}\partial_q z^\nu K_\nu^{(1)}((1-q^2)z;q^2)=\frac{q^{-\nu^2+1/2}}{4(a_\nu a_{-\nu})^{3/2}\sin\nu\pi}\left[a_\nu\frac{2}{(1+q)z}\partial_q z^\nu I_{-\nu}^{(1)}-a_{-\nu}\frac{2}{(1+q)z}\partial_q z^\nu I_\nu^{(1)}\right]=$$

$$=\frac{q^{-\nu^2+1/2}}{4(a_\nu a_{-\nu})^{3/2}\sin\nu\pi}[a_\nu z^{\nu-1}I_{-\nu+1}^{(1)}-a_{-\nu}z^{\nu-1}I_{\nu-1}^{(1)}].$$

As it follows from (43) $a_\nu=a_{\nu-1}q^{-\nu+1/2}$, $\qquad a_{-\nu}=a_{-\nu+1}q^{-\nu+1/2}$. So

$$\frac{2}{(1+q)z}\partial_q z^\nu K_\nu^{(1)}=$$

$$=-z^{\nu-1}\frac{q^{-(\nu-1)^2+1/2}}{4(a_{\nu-1}a_{-\nu+1})^{3/2}\sin(\nu-1)\pi}[a_{\nu-1}z^{\nu-1}I_{-\nu+1}^{(1)}-a_{-\nu+1}z^{\nu-1}I_{\nu-1}^{(1)}]=-z^{\nu-1}K_{\nu-1}^{(1)}.$$

It is easy to see that $K_{-\nu}^{(j)}((1-q^2)z;q^2)=K_\nu^{(j)}((1-q^2)z;q^2)$. Thus from (57)

$$\frac{2}{(1+q)z}\partial_q z^{-\nu}K_\nu^{(1)}=\frac{2}{(1+q)z}\partial_q z^{-\nu}K_{-\nu}^{(1)}=-z^{-\nu-1}K_{-\nu-1}^{(1)}=-z^{-\nu-1}K_{\nu+1}^{(1)}.\blacksquare$$

**Proposition 5.4** *The function* $K_\nu^{(1)}((1-q^2)z;q^2)$ *satisfies the next functional relations*

$$K_{\nu-1}^{(1)}((1-q^2)z;q^2)-K_{\nu+1}^{(1)}((1-q^2)z;q^2)=-\frac{2}{(1-q^2)z}(q^{-\nu}-q^\nu)K_\nu^{(1)}((1-q^2)qz;q^2),$$

$$K_{\nu-1}^{(1)}((1-q^2)z;q^2)+K_{\nu+1}^{(1)}((1-q^2)z;q^2)=-\frac{4}{(1-q^2)z}K_\nu^{(1)}((1-q^2)z;q^2)+$$

$$+\frac{2}{(1-q^2)z}(q^{-\nu}+q^\nu)K_\nu^{(1)}((1-q^2)qz;q^2).$$

**Proof.** Denote the coefficients in front of the square brackets in (53) by $A_\nu$. Then from (43)

$$A_{\nu-1} = -A_\nu q^{-\nu+1/2}, \qquad A_{\nu+1} = -A_\nu q^{\nu+1/2}.$$

The statement follows from (43),(53),(21) and (22).

**Proposition 5.5** *The function $K_\nu^{(2)}((1-q^2)z;q^2)$ satisfies the following relations*

$$\frac{2}{(1+q)z}\partial_q z^\nu K_\nu^{(2)}((1-q^2)z;q^2) = -q^{-\nu+1}z^{\nu-1}K_{\nu-1}^{(2)}((1-q^2)z;q^2),$$

$$\frac{2}{(1+q)z}\partial_q z^{-\nu} K_\nu^{(2)}((1-q^2)z;q^2) = -q^{-\nu+1}z^{-\nu-1}K_{\nu+1}^{(2)}((1-q^2)z;q^2).$$

**Proposition 5.6** *The function $K_\nu^{(2)}((1-q^2)z;q^2)$ satisfies the next functional relations*

$$K_{\nu-1}^{(2)}((1-q^2)z;q^2) - K_{\nu+1}^{(2)}((1-q^2)z;q^2) = -\frac{2q^{\nu-1}}{(1-q^2)z}(q^{-\nu}-q^\nu)K_\nu^{(2)}((1-q^2)qz;q^2),$$

$$K_{\nu-1}^{(2)}((1-q^2)z;q^2) + K_{\nu+1}^{(2)}((1-q^2)z;q^2) = -\frac{4q^{\nu-1}}{(1-q^2)z}K_\nu^{(2)}((1-q^2)z;q^2)+$$

$$+\frac{2q^{\nu-1}}{(1-q^2)z}(q^{-\nu}+q^\nu)K_\nu^{(2)}((1-q^2)qz;q^2).$$

The proof of Propositions 5.5 and 5.6 are the same as 5.3 and 5.4.

**Remark 5.1**

$$\lim_{q\to 1-0} K_\nu^{(j)}((1-q^2)z;q^2) = K_\nu(z), \qquad j = 1,2. \tag{58}$$

Really it follows from (43) and (44) if $q=1$ then $a_\nu$ is independent of $\nu$, and $a_\nu = \frac{1}{\sqrt{2\pi}}$. Now (58) follows from Remark 3.1.

**Remark 5.2** *If $q \to 1-0$ the representations (37) and (55) give us the well known asymptotic decompositions for the functions $I_\nu(z)$ and $K_\nu(z)$ respectively [13].*

## 6 Some relations for the basic hypergeometric functions

Here we summarize some relations for the basic hypergeometric functions that we have already derived in Section 4. It is worthwhile to note that they are based on the identifcations of the power and Laurent expansions for the same $q$-functions. In the limit $q \to 1$ it is became impossible since the later expansion diverges. Remind that

$${}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, \frac{2q}{(1-q^2)z}) = \Phi_\nu(z), \tag{59}$$

$$a_\nu = \sqrt{\frac{2}{1-q^2}}e_q(-1)\frac{I_\nu^{(2)}(2;q^2)}{\Phi_\nu(\frac{2}{1-q^2})},\ \nu \neq n, \quad a_n = \sqrt{\frac{q^{-n^2+1/2}\ln q^{-2}}{2\pi(1-q^2)}}.$$

Assume in(59) $\frac{2q}{(1-q^2)z} = u, \quad |u| < 1$. Then

$$a_{\nu-1}\ {}_2\Phi_1(q^{\nu-1/2}, q^{-\nu+3/2}; -q; q, u) - a_{\nu+1}\ {}_2\Phi_1(q^{\nu+3/2}, q^{-\nu-1/2}; -q; q, u) =$$
$$= a_\nu q^{-1/2}(q^{-\nu} - q^\nu)\,(u/q - 1)\ {}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, u/q),$$
$$a_{\nu-1}\ {}_2\Phi_1(q^{\nu-1/2}, q^{-\nu+3/2}; -q; q, u) + a_{\nu+1}\ {}_2\Phi_1(q^{\nu+3/2}, q^{-\nu-1/2}; -q; q, u) =$$
$$= 2a_\nu u/q\ {}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, u) -$$
$$-a_\nu q^{-1/2}(q^{-\nu} + q^\nu)\,(u/q - 1)\ {}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, u/q).$$

The last relation coming from the q-Wronskian (29) has been proved for the nonineteger $\nu$

$$(u/q + 1)\ {}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, u)\ {}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, -u/q) -$$
$$(u/q - 1)\ {}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, -u)\ {}_2\Phi_1(q^{\nu+1/2}, q^{-\nu+1/2}; -q; q, u/q) = 2.$$

Since ${}_2\Phi_1$ is the continuous function of $\nu$, then this equality is valid for integer $\nu$.

# References


[1] Jackson F.H. The application of basic numbers to Bessel's and Legendre's functions, Proc. London math. Soc. (2) **2** (1905) 192-220

[2] Floreanini R. and Vinet L. "Addition formulas for q-Bessel functions" preprint University of Montreal, UdeM-LPN-TH60, (1991); "Representations of quantum algebras and q-special functions, preprint Universite de Montreal (1991)

[3] Vaksman L. and Korogodskii L., "Algebra of bounded functions on the quantum group of the motions of the plane and q-analogues of Besel functions", Soviet. Math. Dokl. **39** (1989) 173-177

[4] Vilenkin, N.Ya., Special Functions and the Theory of Group Representations, Amer. Math. Soc. Transl. of Math. Monographs **22**, Providence 1968

[5] Dynkin E.B. "Nonegative eigenfunctions of the Laplace-Beltrami operator and the Brownian motion in some symmetric spaces, Dokl. Acad, Nauk **141** (1961) 288-291

[6] Olshanetsky M. "Martin boundaries for real semisimple Lie groups", Journ. of Funct. Anal. **126** (1994) 169-216

[7] Jacquet H. P. B., Fonctions de Whittaker associees aux groupes de Chevalley, Bull.Soc. Math. France, 95 (1967) 243-309

[8] Schiffmann G., Integrales d'entrelacement et fonctions de Whittaker, Bull.Soc.Math.France, 99 (1971) 3-72

[9] Semenov-Tian-Shansky M., Quantization of the open Toda chains, in "Sovremenie problemi matematiki" VINITI, v.16 (1987) 194-226, (in Russian)

[10] Ginsparg P. and Moore G., Lectures on 2d Gravity and 2d Strings, preprint Yale Univ. TCTP-P23 -92

[11] Olshanetsky M. and Rogov V., "Liouville quantum mechanics on a lattice from geometry of quantum Lorentz group ", Journ. Phys. A: Math. Gen, **27** (1994) 4669-4683

[12] Gasper G. and Rahman M., Basic Hypergeometric Series. Cambridge: Cambridge University Press, (1990)

[13] Bateman H., and Erdlyi A., "Higher transcendental functions," v.2 Mc Graw-Hill Book Company, 1953